

# On the generalized Carlitz module.

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**Abstract.** The aim of this note is to gather formal similarities between two apparently different functions; *Euler's function*  $\Gamma$  and *Anderson-Thakur function*  $\omega$ . We discuss these similarities in the framework of the *generalized Carlitz's module*, a common structure which can be helpful in framing the theories of both functions. We further analyze several noticeable differences while investigating the *exponential functions* associated to these structures.

## 1 Introduction

Euler's gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt \quad (1)$$

is known to satisfy several functional relations, among which the so-called *translation formula*, *multiplication formulas* and the *reflection formula*. In a recent work [22], the author used the so-called *Anderson-Thakur function*  $\omega$  (introduced in [4]) to deduce properties of a new class of *deformations of Carlitz's zeta values*, which are special values of zeta functions in the framework of the theory of global fields of positive characteristic. Already in [22], we highlighted several similarities between  $\Gamma$  and  $\omega$ ; in particular, it was observed there that functions closely related to  $\omega$  also satisfy functional relations similar to those satisfied by  $\Gamma$ .

The aim of this expository note is to gather further analogies and highlight differences between these functions by using the *formalism of Carlitz's module* in application to general *difference fields*  $(\mathcal{K}, \tau)$ , that is, fields with a distinguished endomorphism  $\tau$ . The idea of making use of the formalism of difference fields is natural and not new, see for example Mumford's paper [20]. Later, Hellegouarch and his school (see for example [15, 16]), highlighted how several properties of Carlitz module (and more generally of Drinfeld modules) are formal consequences of this approach.

A key tool of this formalism is the notion of *exponential*. In general, the latter is a formal series of the skew ring of formal series  $\mathcal{K}[[\tau]]$ . In many cases, we can associate a function to it. We will discuss the exponential functions underlying respectively the gamma function and the function of Anderson and Thakur (commonly known as the *Carlitz's exponential*) and we will also describe how, although partially submitted to a common formalism, these functions are essentially of different nature.

The paper will also make use of Carlitz's formalism to interpret classical properties of the Hurwitz's zeta function; this will be helpful in a tentative to better understand properties of the above mentioned deformations of Carlitz's zeta values, introduced in [22], and denoted by  $L(\chi_t^\beta, \alpha)$ .

## 2 The generalized Carlitz's module

A *difference field*  $(\mathcal{K}, \tau)$  is a field  $\mathcal{K}$  with a distinguished field endomorphism  $\tau : \mathcal{K} \rightarrow \mathcal{K}$ . A difference field may also be simply denoted by  $\mathcal{K}$ . An extension of difference fields  $i : \mathcal{K} \rightarrow \mathcal{K}'$  (alternative notation  $\mathcal{K}'/\mathcal{K}$ ) is an extension of fields equipped with respective endomorphisms  $\tau, \tau'$  such that  $i\tau = \tau'i$  (see Levin's book [19, Definition 2.1.4] for a slightly more general notion and for the background on difference fields). Every difference field can be embedded in an *inversive difference field*, that is, a difference field such that the distinguished endomorphism is an automorphism (see Cohn's paper [13]; see also Levin, [19, Proposition 2.1.7]). In this case, for all  $n \in \mathbb{Z} \setminus \{0\}$ ,  $(\mathcal{K}, \tau^n)$  again is an inversive difference field.

The *field of constants*  $\mathcal{K}^\tau$  of a difference field  $\mathcal{K}$  is the subfield of elements  $c$  of  $\mathcal{K}$  such that  $\tau(c) = c$ , with  $\tau$  the distinguished endomorphism. The *field of periodic points*  $\mathcal{K}^{\text{per.}}$  of  $\mathcal{K}$  is the subfield of  $\mathcal{K}$  whose elements are the  $c \in \mathcal{K}$  such that, for some  $n \in \mathbb{Z}_{>0}$ , we have  $\tau^n(c) = c$ ; this subfield is the algebraic closure of  $\mathcal{K}^\tau$  in  $\mathcal{K}$ .

From now on, all the difference fields we consider are inversive. The most relevant examples we are interested in, concern the couples  $(\mathcal{K}, \tau)$  where  $\mathcal{K}$  is:

- (1) The field  $F$  of meromorphic functions  $f : \mathbb{C} \rightarrow \mathbb{P}_1(\mathbb{C})$ . In this case, the automorphism

$$\tau : F \rightarrow F$$

is defined, for  $f \in F \setminus \{0\}$ , by

$$\tau(f)(s) = f(s+1).$$

The couple  $(F, \tau)$  is an inversive difference field. We have that  $F^\tau$  is the subfield of periodic meromorphic functions and that  $F^{\text{per.}}$  is the subfield of meromorphic functions which are periodic of integral period, which, as we already pointed out, is algebraically closed in  $F$ .

- (2) The field  $\mathbb{C}_\infty$  defined as follows. Let  $p$  be a prime number,  $q$  a positive power of  $p$  and  $\mathbb{F}_q$  the finite field with  $q$  elements. Then  $\mathbb{C}_\infty$  is the completion of an algebraic closure of the completion  $K_\infty$  of  $K = \mathbb{F}_q(\theta)$  for the  $\theta^{-1}$ -adic valuation. There exists only one absolute value  $|\cdot|$  associated to this valuation, such that  $|\theta| = q$ . The field  $\mathbb{C}_\infty$  is endowed with the automorphism of infinite order  $\tau : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$  defined, for  $c \in \mathbb{C}_\infty$ , by

$$\tau(c) = c^q.$$

The couple  $(\mathbb{C}_\infty, \tau)$  is an inversive difference field. We have  $\mathbb{C}_\infty^\tau = \mathbb{F}_q$  and  $\mathbb{C}_\infty^{\text{per.}} = \mathbb{F}_q^{\text{alg.}}$ , the algebraic closure of  $\mathbb{F}_q$  in  $\mathbb{C}_\infty$ .

In each example, we will try to adopt proper coherent notations. However, this is not always compatible with the need for keeping them simple. We will then accept abuses of notation although the reader will be able to recognize the appropriate structures thanks to a detailed description of the context.

We will mainly focus on these two examples to keep the length of this note reasonable, but many other choices of  $(\mathcal{K}, \tau)$  are interesting; some of them will be occasionally mentioned along this note.

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<sup>1</sup>Here, the variable on which  $\tau$  acts, will be most of the time denoted by  $s$ . However, sometimes, we will need to deal with several variables and will then adopt the following convention. For a variable  $s$ , the *right shift operator* that sends  $s$  to  $s+1$  and keeps all other variables parameters etc. fixed, will be denoted by  $\tau_s$ . For example, we have, for a suitable function of two variables  $f(z, s)$ ,  $\tau_s(f(z, s)) = f(z, s+1)$ .

## 2.1 Definition and first properties

We now closely follow Carlitz [8]. Our discussion also follows quite closely the presentation in Goss' book [14] (see, for example, Proposition 3.3.10). Here, we additionally deal with a general inversive difference field satisfying certain conditions.

Let us fix an inversive difference field  $(\mathcal{K}, \tau)$  as above. We will use the skew ring  $\mathcal{K}[[\tau]]$  of formal series

$$\sum_{i \geq 0} c_i \tau^i, \quad c_i \in \mathcal{K}, \quad (2)$$

with the product defined by the formula

$$\sum_{i \geq 0} c_i \tau^i \cdot \sum_{j \geq 0} d_j \tau^j = \sum_{k \geq 0} \left( \sum_{i+j=k} c_i \tau^i(d_j) \right) \tau^k.$$

The ring  $\mathcal{K}[[\tau]]$  contains the subring  $\mathcal{K}[\tau]$  of *polynomials in  $\tau$* , that is, series (2) such that  $c_i = 0$  for all but finitely many indexes  $i$ . The following construction will look at first sight quite abstract. Eventually, our approach will later become very concrete.

In order to construct generalized Carlitz's modules associated to  $\mathcal{K}$  we need to fulfill the following hypothesis:

*Hypothesis.* We will suppose, in all the following, that  $\mathcal{K} \setminus \mathcal{K}^{\text{per.}} \neq \emptyset$ , or in other words, that  $\mathcal{K}$  is a transcendental extension of  $\mathcal{K}^{\text{per.}}$ , property true for both the Examples (1) and (2) above. This implies that  $\tau$  is of infinite order.

Let us choose, as we can,  $\vartheta \in \mathcal{K}$  transcendental over  $\mathcal{K}^{\text{per.}}$  and let us define, inductively, for  $z \in \mathcal{K}$ :

$$\begin{aligned} E_0^{(\vartheta)}(z) &= z \\ E_1^{(\vartheta)}(z) &= \frac{\tau(E_0^{(\vartheta)}(z)) - E_0^{(\vartheta)}(z)}{\tau(\vartheta) - \vartheta} = \frac{\tau(z) - z}{\tau(\vartheta) - \vartheta} \\ E_2^{(\vartheta)}(z) &= \frac{\tau(E_1^{(\vartheta)}(z)) - E_1^{(\vartheta)}(z)}{\tau^2(\vartheta) - \vartheta} \\ &\vdots \\ E_k^{(\vartheta)}(z) &= \frac{\tau(E_{k-1}^{(\vartheta)}(z)) - E_{k-1}^{(\vartheta)}(z)}{\tau^k(\vartheta) - \vartheta} \\ &\vdots \end{aligned}$$

To avoid heavy notations, we will drop the superscript  $(\cdot)^{(\vartheta)}$ . This construction is borrowed from Carlitz [8], see also [14, Section 3.5] and [16, Section 3].

It is easy to see that the sequence of operators  $(E_k)_{k \geq 0}$  is a  $\tau$ -linear *higher derivation* on  $\mathcal{K}$ . In other words, for all  $k \geq 0$ ,  $E_k$  is  $\mathcal{K}^\tau$ -linear and if  $x, y$  are elements of  $\mathcal{K}$ , the following  $\tau$ -twisted version of Leibniz rule holds, for  $k \geq 0$ :

$$E_k(xy) = \sum_{i+j=k} E_i(x) \tau^i(E_j(y)). \quad (3)$$

The next definition is then meaningful:

**Definition 1** The *generalized Carlitz module* associated to  $(\mathcal{K}, \tau)$  and  $\vartheta$  is the injective  $\mathcal{K}^\tau$ -algebra homomorphism

$$\phi_{\mathcal{K}, \tau, \vartheta} : \mathcal{K} \rightarrow \mathcal{K}[[\tau]]$$

defined, for  $z \in \mathcal{K}$ , by

$$\phi_{\mathcal{K}, \tau, \vartheta}(z) = \sum_{k \geq 0} (-1)^k E_k^{(\vartheta)}(z) \tau^k.$$

We notice that, independently on the choice of  $(\mathcal{K}, \tau)$  and  $\vartheta$ ,

$$\phi_{\mathcal{K}, \tau, \vartheta}(\vartheta) = \vartheta - \tau.$$

From now on, we will write  $\phi$  instead of  $\phi_{\mathcal{K}, \tau, \vartheta}$  to simplify notations, if there is no risk of confusion.

**Lemma 2** For  $n \geq 2$ , let  $z_1, \dots, z_n$  be elements of  $\mathcal{K}$ . We have:

$$\det \begin{pmatrix} E_0(z_1) & \dots & E_0(z_n) \\ \vdots & & \vdots \\ E_{n-1}(z_1) & \dots & E_{n-1}(z_n) \end{pmatrix} = 0$$

If and only if  $z_1, \dots, z_n$  are  $\mathcal{K}^\tau$ -linearly dependent.

*Proof.* By induction on  $n \geq 2$ , one proves the identity

$$\det \begin{pmatrix} E_0(z_1) & \dots & E_0(z_n) \\ \vdots & & \vdots \\ E_{n-1}(z_1) & \dots & E_{n-1}(z_n) \end{pmatrix} = F_n^{-1} \det \begin{pmatrix} z_1 & \dots & z_n \\ \vdots & & \vdots \\ \tau^{n-1}(z_1) & \dots & \tau^{n-1}(z_n) \end{pmatrix}, \quad (4)$$

where

$$F_n = \prod_{i=0}^{n-1} \prod_{j=1}^{n-1-i} (\tau^{i+j} \vartheta - \tau^i \vartheta).$$

Indeed, we can inductively replace the definition of  $E_k(z_i)$  ( $i = 1, \dots, n$ ) in the following way, where

we have set  $G_i = \tau^i \prod_{j=1}^{n-1-i} (\tau^j \vartheta - \vartheta)$  for  $i = 0, \dots, n-1$  (so that  $F_n = \prod_{i=0}^{n-1} G_i$ ):

$$\begin{aligned}
\det \begin{pmatrix} E_0(z_1) & \dots & E_0(z_n) \\ E_1(z_1) & \dots & E_1(z_n) \\ \vdots & & \vdots \\ E_{n-1}(z_1) & \dots & E_{n-1}(z_n) \end{pmatrix} &= \det \begin{pmatrix} z_1 & \dots & z_n \\ \frac{\tau z_1 - z_1}{\tau \vartheta - \vartheta} & \dots & \frac{\tau z_n - z_n}{\tau \vartheta - \vartheta} \\ \vdots & & \vdots \\ \frac{\tau E_{n-2}(z_1) - E_{n-2}(z_1)}{\tau^{n-1} \vartheta - \vartheta} & \dots & \frac{\tau E_{n-2}(z_n) - E_{n-2}(z_n)}{\tau^{n-1} \vartheta - \vartheta} \end{pmatrix} \\
&= G_0^{-1} \det \begin{pmatrix} z_1 & \dots & z_n \\ \tau z_1 & \dots & \tau z_n \\ \tau E_1(z_1) & \dots & \tau E_1(z_n) \\ \vdots & & \vdots \\ \tau E_{n-2}(z_1) & \dots & \tau E_{n-2}(z_n) \end{pmatrix} \\
&= G_0^{-1} G_1^{-1} \det \begin{pmatrix} z_1 & \dots & z_n \\ \tau z_1 & \dots & \tau z_n \\ \tau^2 z_1 & \dots & \tau^2 z_n \\ \tau^2 E_1(z_1) & \dots & \tau^2 E_1(z_n) \\ \vdots & & \vdots \\ \tau^2 E_{n-3}(z_1) & \dots & \tau^2 E_{n-3}(z_n) \end{pmatrix} = \dots
\end{aligned}$$

and so on, hence yielding the identity (4).

Since  $F_n$  is non-zero by the assumption that  $\vartheta \notin \mathcal{K}^{\text{per}}$ , the Lemma follows from (4) and the Wronskian criterion <sup>(2)</sup>. See also [14, Lemma 1.3.3].  $\square$

In all the following, we denote by  $\mathcal{A}$  the ring  $\mathcal{K}^\tau[\vartheta] \subset \mathcal{K}$ . Since

$$\phi(\vartheta) = \vartheta - \tau,$$

we observe that  $\phi$  induces a  $\mathcal{K}^\tau$ -algebra homomorphism

$$\phi : \mathcal{A} \rightarrow \mathcal{A}[\tau].$$

For example,

$$\begin{aligned}
\phi(\vartheta^2) &= \vartheta^2 - (\vartheta + \tau(\vartheta))\tau + \tau^2, \\
\phi(\vartheta^3) &= \vartheta^3 - (\tau(\vartheta^2) + \vartheta\tau(\vartheta) + \vartheta^2)\tau + (\tau^2(\vartheta) + \tau(\vartheta) + \vartheta)\tau^2 - \tau^3.
\end{aligned}$$

In fact, more is true, as the following elementary proposition shows.

**Proposition 3** *Let  $a$  be an element of  $\mathcal{K}$ , so that  $\phi(a) \in \mathcal{K}[[\tau]]$ . We have that  $\phi(a) \in \mathcal{K}[\tau]$  if and only if  $a \in \mathcal{A}$ .*

*Proof.* Assume first that  $a \in \mathcal{A} = \mathcal{K}^\tau[\vartheta]$ . Then, we may write

$$a = \sum_{i=0}^d a_i \vartheta^i, \quad a_i \in \mathcal{K}^\tau, \quad a_d \neq 0.$$

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<sup>2</sup>The determinants above can also be viewed as generalizations of so-called *Moore's determinants* (see [14, Section 1.3]); sometimes, they are called *casoratians*.

We then deduce that

$$\phi(a) = a + \cdots + (-1)^d a_d \tau^d,$$

agreeing with our statement.

Let us now assume that  $\phi(a) \in \mathcal{K}[\tau]$ . Then, there exists  $d'$  such that

$$\phi(a), \phi(1), \phi(\vartheta), \dots, \phi(\vartheta^{d'})$$

are  $\mathcal{K}$ -linearly dependent operators of  $\mathcal{K}[\tau]$ . By Lemma 2,  $a, 1, \vartheta, \dots, \vartheta^{d'}$  are linearly dependent over  $\mathcal{K}^\tau$  so that  $a \in \mathcal{K}^\tau[\vartheta]$ .  $\square$

We deduce:

**Corollary 4** *For any choice of  $\vartheta \in \mathcal{K} \setminus \mathcal{K}^{\text{per.}}$ , the Carlitz's module induces an injective  $\mathcal{K}^\tau$ -algebra homomorphism*

$$\phi : \mathcal{A} \rightarrow \mathcal{A}[\tau]$$

such that, for all  $a \in \mathcal{A}$ , monic of degree  $d$  in  $\vartheta$ ,  $\phi(a) = a + \cdots + (-1)^d \tau^d$ .

*Link with Example (1).* We choose  $\vartheta = \text{Id}$ , the identity map  $\mathbb{C} \rightarrow \mathbb{C}$ . This is licit: since  $F^{\text{per.}}$  is algebraically closed in  $F$  and  $\text{Id}$  is not periodic, it is transcendental over  $F^{\text{per.}}$ . In the following, we will identify  $\text{Id}$  with the variable  $s$  of our meromorphic functions in  $F$  (yet another abuse of notation). Therefore, we will write  $\vartheta = s$ .

We recall that for  $a \in F$ ,  $\phi(a) \in F[\tau]$  if and only if  $a \in F^\tau[s]$ . In particular:

$$\phi(s) = s\tau^0 - \tau^1$$

and, for  $a \in \mathbb{C}[s]$  of degree  $d$ ,

$$\phi(a) = \sum_{i=0}^d E_i(a) \tau^i,$$

where  $E_i(a)$  is a polynomial of degree  $d-i$  in  $s$  with complex coefficients. For example, one has:

$$\begin{aligned} \phi(s^2) &= s^2 - (2s+1)\tau + \tau^2 \\ \phi(s^3) &= s^3 - (3s^2+3s+1)\tau + 3(1+s)\tau^2 - \tau^3. \end{aligned}$$

More generally, one checks easily that if  $a$  is of degree  $d$  and  $c$  is its leading coefficient, then  $E_d(a) = (-1)^d c$ . Our definition allows to evaluate  $\phi$  at elements of  $\mathcal{K} \setminus \mathcal{A}$  as well. For example, it is easy to show that:

$$\phi(s^{-1}) = \sum_{j \geq 0} (-1)^{j+1} (s)_{j+1}^{-1} \tau^j,$$

where  $(s)_j = s(s-1) \cdots (s-j+1)$  denotes the  $j$ -th Pochhammer polynomial in  $s$ .

*Link with Example (2).* If  $\vartheta = \theta$ , we shall write, in all the following,  $A = \mathcal{A} = \mathbb{F}_q[\theta]$ . Let us suppose that  $\vartheta = \theta$ , by Lemma 2, that  $\phi(a) \in \mathbb{C}_\infty[\tau]$  if and only if  $a \in A$ . More precisely, we have:

$$\phi(a) = \sum_{i=0}^d (-1)^i E_i^{(\theta)}(a) \tau^i, \tag{5}$$

where  $d = \deg_\theta(a)$ . In particular,

$$\phi(\theta) = \theta - \tau,$$

and we obtain the *old* module originally considered by Carlitz, for example in his papers [8, 9, 10, 11].

**Important note.** However, to ease the connection with the more recent literature, we will use, from now on in any reference to the Example (2), the *actual* Carlitz's module, that is, the  $\mathbb{F}_q$ -algebra homomorphism

$$\phi : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty[[\tau]]$$

determined, for  $z \in \mathcal{K}$ , by

$$\phi(z) = \sum_{k \geq 0} E_k^{(\vartheta)}(z) \tau^k. \quad (6)$$

In particular for  $\vartheta = \theta$ ,

$$\phi(\theta) = \theta + \tau.$$

This makes our exposition slightly more complicated but should not lead to confusion. In all references to the Example (2) (including when  $\vartheta \neq \theta$ ), Carlitz's module will be defined by (6) instead of (5). It would be also possible to switch all our discussion by considering (6) to define Carlitz's module for *all* inversive difference fields  $\mathcal{K}$  as above, but, in this case, we would lose the direct connection with the theory of the gamma function in the framework of Example (1), as the reader will notice reading the rest of this paper. Also, we are convinced that Carlitz was aware of these analogies in choosing (5) for his definition and this led us to make our choice for the present paper.

## 2.2 Torsion

**Definition 5** Let  $a$  be an element of  $\mathcal{A} = \mathcal{K}^\tau[\vartheta]$  and  $x$  an element of  $\mathcal{K}$ . We say that  $x \in \mathcal{K}$  is an element of  $a$ -torsion if  $\phi(a)(x) = 0$ .

The subset  $\ker(\phi(a))$  of elements of  $a$ -torsion of  $\mathcal{K}$  is easily seen to be a  $\mathcal{K}^\tau$ -vector space of finite dimension  $\leq d = \deg_\vartheta(a)$  and, at once, an  $\mathcal{A}$ -module, because if  $\phi(a)(x) = 0$  then, for all  $b \in \mathcal{A}$ ,  $0 = \phi(b)\phi(a)(x) = \phi(a)(\phi(b)(x))$ .

In the framework of Example (1), we have the following result.

**Proposition 6** Let  $f$  be an element of  $\mathcal{A} = F^\tau[s]$ . The set  $\ker(f)$  is a  $F^\tau$ -vector subspace of  $F$  of dimension  $d$ , where  $d$  is the degree in  $s$  of  $f$ .

*Proof.* This follows from a more general result due to Praagman, [23, Theorem 1, p. 102] which says that, with  $\tau$  as in Example (1), a linear homogeneous  $\tau$ -difference equation of order  $d$  with coefficients in  $F$  can be fully solved in  $F$ , and the set of solutions is an  $F^\tau$ -vector subspace of  $F$  of dimension  $d$ .  $\square$

In the framework of Example (2), we have the following well known analogue of Proposition 6.

**Proposition 7** Let  $a$  be an element of  $A = \mathbb{F}_q[\theta]$  of degree  $d$  in  $\theta$ . The set  $\ker(a)$  is an  $\mathbb{F}_q$ -vector subspace of  $K^{\text{sep}}$  of dimension  $d$ .

We recall that  $K^{\text{sep}}$  denotes the maximal separable extension of  $K = \mathbb{F}_q(\theta)$  in  $\mathbb{C}_\infty$ .

*Proof.* Let  $a$  be as in the statement of the Proposition. Solving the  $\tau$ -linear equation  $\phi(a)X = 0$  amounts to solve a separable algebraic equation of degree  $q^d$  the set of solutions of which is an  $\mathbb{F}_q$ -vector space of dimension  $d$ .  $\square$

In general, for a given  $a \in \mathcal{A}$ , the existence of non-zero torsion elements is not guaranteed and reflects the choice of the field  $\mathcal{K}$ . The dimension of the vector space  $\ker(a)$  may be, in certain circumstances, strictly smaller than  $d$ . But up to extension the difference field  $\mathcal{K}$ , we may assume that the dimension is exactly  $d$ .

Indeed, there is a procedure of construction of an extension of  $\mathcal{K}$  in which all the spaces  $\ker(a)$  have maximal dimension, which is similar in spirit to the construction of an algebraically closed extension of a field. Generalizing algebraic equations in one indeterminate, a  $\tau$ -difference equation defined over  $\mathcal{K}$  is a formal identity

$$P(X, \tau X, \dots, \tau^l X) = 0, \quad (7)$$

where  $P$  is a non-zero polynomial in  $n$  indeterminates. Solving the above  $\tau$ -difference equation amounts to look for the  $(l+1)$ -tuples

$$(x_0, x_1, \dots, x_l)$$

with coordinates in some difference field extension of  $\mathcal{K}$  such that  $P(x_0, \dots, x_l) = 0$ , and such that  $x_i = \tau^i x_0$  for all  $i$ . Any  $x$  with the above properties is called a *solution* of the  $\tau$ -difference equation (7).

By the so-called ACFA theory of Chatzidakis-Hrushowski [12], there exists an *existentially closed* difference field extension  $(\mathbb{K}, \tau)$  of  $(\mathcal{K}, \tau)$ . We recall here that an existentially closed difference field  $(\mathbb{K}, \tau)$  is characterized by the property that every finite system of  $\tau$ -difference equations defined over  $\mathbb{K}$  having a solution in some field extension of  $\mathbb{K}$  already has a solution in this field. An existentially closed extension  $\mathbb{K}$  of  $\mathcal{K}$  is, in particular, algebraically closed, and such that, for all  $n > 0$ ,

$$\mathbb{K}^{\tau^n} \cap \mathcal{K} = \mathcal{K}^{\tau^n},$$

so that  $\mathbb{K}^{\text{per}} \cap \mathcal{K} = \mathcal{K}^{\text{per}}$ . There is no canonical choice of such an extension; this is due to the fact that in general, given two difference field extensions  $\mathcal{L}$  and  $\mathcal{M}$  of a given difference field  $(\mathcal{K}, \tau)$ , a good notion of “compositum” (a minimal difference field extension of  $\mathcal{K}$  in which  $\mathcal{L}$  and  $\mathcal{M}$  can be simultaneously embedded) is not always available. This yields the notion of *compatible* and *non-compatible* difference field extensions. See [19, Chapter 5].

The following Proposition holds.

**Proposition 8** *Let us assume that  $(\mathbb{K}, \tau)$  is existentially closed. If  $L = a_0\tau^0 + \dots + a_d\tau^d \in \mathbb{K}[\tau]$  is such that  $a_d a_0 \neq 0$ , then there exist  $x_0, \dots, x_{d-1} \in \mathbb{K}$ ,  $\mathbb{K}^\tau$ -linearly independent, such that  $L(x_i) = 0$  for  $i = 0, \dots, d-1$ . Moreover, the set of solutions of the linear  $\tau$ -difference equation  $L(X) = 0$  is equal to the  $\mathbb{K}^\tau$ -vector space of dimension  $d$  generated by  $x_0, \dots, x_{d-1}$ .*

*Sketch of proof.* This result is well known, see [19, Proposition 8.2.4]. It is easy to show that any non-trivial linear  $\tau$ -difference equation with coefficients in  $\mathbb{K}$  has a non-trivial solution in  $\mathbb{K}$ . We proceed by induction on  $d \geq 0$ . If  $d = 0$ , the statement is trivial. Let us assume now that  $d > 0$ . Since  $\mathbb{K}$  is existentially closed, there exists a solution  $x_0 \neq 0$  of  $L(x) = 0$ . Right division



algorithm holds in  $\mathbb{K}[\tau]$ , so that there exists  $\tilde{L} \in \mathbb{K}[\tau]$  unique, with  $L = \tilde{L}L_{x_0}$ , where, for  $x \in \mathbb{K}^\times$ , we have written  $L_x = \tau - (\tau x)/x$ . Since the order of the difference operator  $\tilde{L}$  is  $d-1$ , there exist  $y_1, \dots, y_{d-1}$ ,  $\mathbb{K}^\tau$ -linearly independent elements of  $\mathbb{K}$  such that  $\tilde{L}y_i = 0$  for all  $i$ . Now, for all  $i \geq 1$ , let  $x_i$  be a solution of  $L_{x_0}(x_i) = y_i$  (they exist, again because  $\mathbb{K}$  is existentially closed). Then,  $x_0, x_1, \dots, x_{d-1}$  are  $d$  linearly independent elements of  $\mathbb{K}$ , solutions of  $L(x) = 0$  such that the set of the solutions in  $\mathbb{K}$  of the equation  $L(x) = 0$ , a  $\mathbb{K}$ -vector space, has dimension  $\geq d$ . Now, it is easy to verify, by using the Wronskian Lemma, that the dimension is exactly  $d$ .  $\square$

We observe that for all  $a \in \mathcal{A}$  of degree  $d$  in  $\vartheta$ ,  $\phi(a)$  is precisely of the form  $L$  as in Proposition 8. Therefore, by using Proposition 3 and a little computation, we deduce the following Lemma.

**Lemma 9** *Let  $a \in \mathcal{A}$  be of degree  $d$  in  $\vartheta$ . If  $\mathcal{K}$  is existentially closed, the vector space  $\ker(\phi(a))$  has dimension  $d$ . Moreover, there is an isomorphism of  $\mathcal{A}$ -modules  $\ker(a) \cong \mathcal{A}/a\mathcal{A}$ .*

We notice that the difference field extension of  $\mathcal{K}$  obtained by adjoining the torsion elements is uniquely determined up to difference field isomorphism.

**Definition 10** A sequence  $(x_i)_{i \geq 1}$  of elements of  $\mathcal{K}$  is  $\vartheta$ -coherent, or coherent, if  $x_1 \neq 0$  with  $\phi(\vartheta)(x_1) = 0$ , and if  $\phi(\vartheta)(x_i) = x_{i-1}$  for all  $i > 1$ . Sometimes, for a given coherent sequence  $\Xi = (x_i)_{i \geq 1}$ , we will need to set  $x_0 = 0$ .

We have the following.

**Lemma 11** *Let  $\Xi = (x_i)_{i \geq 1}$  be a coherent sequence. Then the coefficients  $x_1, x_2, \dots, x_n, \dots$  are linearly independent over  $\mathcal{K}^\tau$ .*

*Proof.* Let us suppose by contradiction that for some  $a_1, \dots, a_m \in \mathcal{K}^\tau$  with  $a_m \neq 0$ , we have  $\sum_{i=1}^m a_i x_i = 0$ . Then, evaluating  $\phi(\vartheta)$  on the left and of the right of this identity and noticing that  $\phi(\vartheta)(a_i x_i) = a_i \phi(\vartheta)(x_i)$ , we find  $a_2 x_1 + \dots + a_m x_{m-1} = 0$ . More generally, evaluating  $\phi(\vartheta^j)$ , we obtain at once

$$\sum_{i=j}^m a_i x_{i-j+1} = 0, \quad j = 1, \dots, m.$$

The vector  $(x_1, \dots, x_m)$  is then the unique, trivial solution of a non-singular homogeneous linear system, a contradiction because we have supposed that  $x_1 \neq 0$ .  $\square$

We consider a new indeterminate  $t$  and the field  $\mathcal{L} = \mathcal{K}((t))$ . We extend  $\tau$  to  $\mathcal{L}$  by setting

$$\tau \left( \sum_{i \geq i_0} c_i t^i \right) = \sum_{i \geq i_0} \tau(c_i) t^i, \quad c_i \in \mathbb{K}, \quad i_0 \in \mathbb{Z}.$$

Then,  $\mathcal{L}$  is again an inversive difference field such that, for all  $n > 0$ ,  $\mathcal{L}^{\tau^n} = \mathcal{K}^{\tau^n}((t))$ .

**Definition 12** Let  $\Xi = (x_i)_{i \geq 1}$  be a coherent sequence of elements of  $\mathcal{K}$ . The *Akhiezer-Baker series* associated to  $\Xi$  is the following element of  $\mathcal{K}[[t]]$ :

$$\omega_{\mathcal{K}, \tau, \vartheta, \Xi}(t) = \sum_{i \geq 0} x_{i+1} t^i.$$

To simplify our notations, we will write  $\omega_\Xi$  instead of  $\omega_{\mathcal{K}, \tau, \vartheta, \Xi}$  when the reference to the data  $\mathcal{K}, \tau, \vartheta$  is clearly indicated.

If  $a = a(\vartheta)$  is an element of  $\mathcal{A}$ , we will write  $a(t)$  for the function of the variable  $t$  or the element of  $\mathcal{L}$  obtained by formal replacement of  $\vartheta$  with  $t$ , which is meaningful because  $\vartheta$  is transcendental over  $\mathcal{K}^\tau$ . The following Proposition justifies the adopted terminology and is easy to prove.

**Proposition 13** *Let  $\Xi = (x_i)_{i \geq 1}$  be a sequence of  $\mathcal{K}$  with  $x_1 \neq 0$  and let us write  $\omega_\Xi = \sum_{i \geq 0} x_{i+1} t^i$ . The following properties are equivalent.*

1.  $\Xi$  is coherent,
2.  $\phi(\vartheta)(\omega_\Xi) = t\omega_\Xi$ ,
3. For all  $a \in \mathcal{A}$ ,  $\phi(a)\omega_\Xi = a(t)\omega_\Xi$ .

In particular, if  $\Xi, \Xi'$  are two coherent sequences, then

$$\omega_\Xi = \lambda_{\Xi, \Xi'} \omega_{\Xi'},$$

where  $\lambda_{\Xi, \Xi'}$  is an element of  $\mathcal{L}^\tau$  determined by  $\Xi, \Xi'$ . A coherent sequence can be always found in an appropriate extension of the difference field  $\mathcal{K}$ . If  $\mathcal{K}$  contains a coherent sequence, then it contains all the coherent sequences.

*Sketch of proof.* The equivalence of points 1 and 2 follows from the identities

$$\phi(\vartheta)(\omega_\Xi) = \sum_{i \geq 0} \phi(\vartheta)(x_{i+1}) t^i, \quad t\omega_\Xi = \sum_{i \geq 1} x_i t^i.$$

The point 3 clearly implies the point 2. The opposite implication is obtained observing that if  $a = a_0 + a_1 \vartheta + \dots + a_d \vartheta^d$  with  $a_0, \dots, a_d \in \mathcal{K}^\tau$ , then  $\phi(a)\omega_\Xi = a_0 \omega_\Xi + a_1 \phi(\vartheta)\omega_\Xi + \dots + a_d \phi(\vartheta^d)\omega_\Xi$ , so that  $\phi(a)\omega_\Xi = a(t)\omega_\Xi$ . The remaining parts of the Proposition follow from the fact that the set of solutions in  $\mathcal{L}$  of the difference equation  $\phi(\vartheta)X = tX$  is either trivial, either a  $\mathcal{L}^\tau$ -vector space of dimension 1.  $\square$

In particular, if  $\mathcal{K}$  is existentially closed, it contains all the coherent sequences.

**Remark 14** The link with the classical Akhiezer-Baker functions of Krichever's axiomatic approach [18] lies in the fact that, just as the latter, the functions  $\omega_{\mathcal{K}, \tau, \vartheta, \Xi}$  are eigenfunctions of the full set of operators  $\phi_{\mathcal{K}, \tau, \vartheta}(a) \in \mathcal{K}[[\tau]]$ , with  $a \in \mathcal{A}$ . Akhiezer-Baker functions are usually defined over Riemann surfaces and are characterized by certain essential singularities. Baker, in [7], noticed the possibility to relate them to the theta series associated to the surface, and Akhiezer pointed out that they can also be constructed, in certain cases, as eigenfunctions of differential operators of order two.

In order to make the link with the classical theory and our choice of terminology clearer we give here a suggestive example of a very particular family of Akhiezer-Baker functions, eigenfunctions of a Lamé operator. The example we give is treated by N. I. Akhiezer in his book [2, Chapter 11] (see also Krichever's paper [17]).

Let  $E$  be a complex elliptic curve with Weierstrass' model  $y^2 = 4x^3 - g_2x - g_3$ , analytically isomorphic to a complex torus  $\mathbb{C}/\Lambda$ , where  $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$  is its lattice of periods. Let  $\wp$  denote the Weierstrass function associated to  $\Lambda$ , so that, for  $\nu \notin \Lambda$ ,  $x = \wp'(\nu)$  and  $y = \wp(\nu)$ . We also denote by  $\zeta$  the Weierstrass zeta function of  $\Lambda$  and by  $\sigma$  the Weierstrass sigma function of  $\Lambda$ . Let us choose

a point  $(x^*, y^*) \in E(\mathbb{C})$  with  $(x^*, y^*) = (\wp(\nu^*), \wp'(\nu^*))$ , for some  $\nu^* \in \mathbb{C} \setminus \Lambda$  and let us additionally consider a complex parameter  $z$ . The Akhiezer-Baker function associated to the datum  $(x^*, y^*, z)$  is the function of the variables  $(x, y) \in E(\mathbb{C})$  depending on the parameter  $z$  defined by

$$\Omega(x, y, z) = \frac{\sigma(\nu + \nu^* - z)e^{z\zeta(\nu)}}{\sigma(\nu - \nu^*)\sigma(\nu^* + z)}.$$

The reader can check, with the elementary properties of  $\zeta$ ,  $\sigma$  in mind, that the above function is a well defined meromorphic function on  $E(\mathbb{C}) \setminus \{\infty\}$  but is not an elliptic function, the obstruction being an (unique) essential singularity at infinity (that is, at  $\nu \equiv 0 \pmod{\Lambda}$ ). By the way, it also has a pole at  $(x^*, y^*)$ . So far, we have looked at this function in the dependence of the variables  $(x, y)$ . Now, one verifies easily, with the help of the classical differential properties of basic Weierstrass functions, that  $\Omega(x, y, z)$ , as a function of the variable  $z$ , for any choice of  $x, y, x^*, y^*$ , is eigenfunction of the Lamé differential operator  $\frac{\partial^2}{\partial z^2} - 2\wp(z)$  with eigenvalue  $-\wp(\nu)$ , namely:

$$\left(\frac{\partial^2}{\partial z^2} - 2\wp(z)\right)\Omega(x, y, z) = -\wp(\nu)\Omega(x, y, z).$$

Therefore, this function is eigenfunction of all the operators in the image of the Krichever module uniquely determined by the above operator, and the corresponding eigenvalues are obtained from the constant terms of the operators, with the variable  $z$  replaced by the parameter  $\nu$  (upon certain necessary normalizations). In analogy with our semi-character  $a(\theta) \mapsto a(t)$ , this suggests the terminology adopted in the present paper.

### 2.3 The gamma function as a torsion element

The function  $\Gamma$  is traditionally defined as the Mellin transform (1) of the function  $e^{-z}$ . Since the relation  $\Gamma(s+1) - s\Gamma(s) = 0$  holds, this function belongs to the  $F^\tau$ -vector space of  $s$ -torsion for the generalized Carlitz module  $\phi$  associated to the difference field  $(F, \tau)$ . More generally, we have the following proposition.

**Proposition 15** *The sequence <sup>(3)</sup>*

$$\Xi = \left( (-1)^{i-1} \frac{\Gamma^{(i-1)}}{(i-1)!} \right)_{i \geq 1}$$

*is coherent. For all  $k \geq 1$ , the  $F^\tau$ -vector space  $\ker(\phi(s^k))$  of the solutions  $X$  of the linear  $\tau$ -difference equation*

$$\phi(s^k)(X) = 0$$

*has dimension  $k$  and is spanned by the  $F^\tau$ -linearly independent functions*

$$\Gamma, \Gamma', \dots, \Gamma^{(k-1)}.$$

*Proof.* Let  $f$  be a function of  $F$  and  $a = a(s)$  and element of  $F^\tau[s]$ . Then we can formally compute, with the rule  $\frac{d}{ds}\tau = 0$  and for  $n > 0$ ,

$$\frac{d}{ds}(\phi(a^n)f(s)) = n \left( \frac{d}{ds}\phi(a) \right) \phi(a^{n-1})f(s) + \phi(a^n)\frac{d}{ds}f(s).$$

---

<sup>3</sup>The notation  $f^{(n)}(s)$  denotes, all along this paper, the derivative  $\frac{d^n}{ds^n}f(s)$ .

Since

$$\frac{d}{ds}\phi(s) = 1,$$

we have in particular that

$$\frac{d}{ds}(\phi(s^n)f(s)) = n\phi(s^{n-1})f(s) + \phi(s^n)\frac{d}{ds}f(s)$$

and if  $f(s)$  is  $s^n$ -torsion, that is,  $\phi(s^n)f(s) = 0$ , we get

$$\frac{\phi(s^n)f'(s)}{n} = -\phi(s^{n-1})f(s).$$

By induction on  $n$  we then see that if  $\phi(s)f = 0$ , then  $(-1)^n \frac{f^{(n)}}{n!} \in \ker(\phi(s^{n+1}))$ . On the other hand, it is easy to verify the following formal identity:

$$\frac{1}{m!} \left( \frac{d}{ds} \right)^m \phi(s^n) = \binom{n}{m} \phi(s^{n-m}),$$

from which one deduces, applying the operator  $(d/ds)^n$  on both sides of the identity  $\phi(s)f = 0$ , that  $\phi(s)f^{(n)} = n f^{(n-1)}$ . The above discussion holds if for example  $f = \Gamma$ , which yields the first properties of the sequence  $\Xi$ . Notice that the linear independence of the coefficients of  $\Xi$  can be also verified directly by comparing the orders of the poles of the functions  $\Gamma^{(n)}$ . In particular,  $F$  contains all the coherent sequences for  $\phi_{F,\tau,s}$ .  $\square$

In the present framework, we have, after Proposition 13:

**Proposition 16** *The Akhiezer-Baker function associated to the coherent sequence  $\Xi$  of Proposition 15 is the formal series*

$$\omega_\Xi(t) = \Gamma(s-t) = \sum_{k \geq 0} (-1)^k \frac{\Gamma^{(k)}(s)}{k!} t^k.$$

One can further verify that the series above is convergent for complex numbers  $s, t$  such that  $|t| < |s| < 1$ ,  $\Re(s) > 0$ .

**Remark 17** An interesting property following from Proposition 16 is that the function

$$\psi(s-t) = \frac{\Gamma'(s-t)}{\Gamma(s-t)} = \sum_{n \geq 0} (-1)^n \frac{\psi^{(n)}(s)}{n!} t^n,$$

where  $\psi^{(n)}(s) = (d/dz)^{n+1} \log \Gamma(s)$  <sup>(4)</sup> is the  $n$ -th polygamma function, is a solution of the difference equation

$$\tau X - X = \frac{1}{s-t}. \tag{8}$$

---

<sup>4</sup>Here  $\log$  denotes the principal determination of the classical logarithm.

**Remark 18** We mention that the torsion of the generalized Carlitz's module is related to classical functions also in the framework of other difference fields  $(\mathcal{K}, \tau)$ . The next is an important example that the reader can find of parallel interest (we might refer to it as to the *Example (3)*; see also [16, pp. 350-354]). Consider  $\mathcal{K}$  the field of meromorphic functions over  $\mathbb{C}^\times$  of which the variable is denoted by  $x$ ,  $q$  a non-zero complex number such that  $|q| > 1$  and  $\tau$  the automorphism of  $\mathcal{K}$  defined by  $\tau x = qx$ , so that  $\mathcal{K}^\tau$  is the field of elliptic functions over the elliptic curve  $\mathbb{C}^\times/q^\mathbb{Z}$ . Then, Carlitz's formalism applies to  $(\mathcal{K}, \tau)$  and it turns out that *Jacobi's theta series*:

$$x_1 = \sum_{m \in \mathbb{Z}} q^{-\frac{m(m+1)}{2}} x^m$$

represents a generator of the  $x$ -torsion. More generally, one can prove that, by setting

$$d_m^{[n]} = q^{-\frac{1}{2}m(m+2n+1)} \prod_{i=1}^n \frac{q^{i+m} - 1}{q^i - 1}$$

and

$$x_{n+1} = \sum_{m \in \mathbb{Z}} d_m^{[n]} x^m, \quad n \geq 0,$$

the sequence  $\Xi = (x_n)_{n \geq 0}$  (with  $x_0 = 0$ ) is coherent, from which one obtains explicitly the corresponding Akhiezer-Baker function  $\omega_\Xi$ .

## 2.4 Some elements of $\ker(\phi(f))$ for $f \in F^\tau[s]$

Let  $f$  be an element of  $F^\tau[s]$ . We have seen in Proposition 6 that the set  $\ker(f)$  is a  $F^\tau$ -vector subspace of  $F$  of dimension  $\deg_s(f)$ . Here, we give some examples of elements of  $\ker(f)$ .

By Proposition 16, for all  $f \in F^\tau[s]$ ,

$$\phi(f)\Gamma(s-t) = f(t)\Gamma(s-t).$$

If  $f \in \mathbb{C}[s]$ , and if  $x \in \mathbb{C}$  is a root of  $f$ , then, obviously,

$$\phi(f)\Gamma(s-x) = f(x)\Gamma(s-x) = 0,$$

and the function  $\Gamma(s-x)$ , meromorphic on  $\mathbb{C}$ , lies in the kernel of  $\phi(f)$ . This simple argument can be deduced from the case of  $f$  of degree one in  $s$ . Indeed,  $\Gamma(s-x)$  obviously generates the kernel of  $\phi(s-x)$  (one uses that the field  $\mathbb{C}$  is algebraically closed).

*Example.* The functions  $\Gamma(s-i)$  and  $\Gamma(s+i)$  generate the two-dimensional kernel of  $\phi(s^2+1)$ .

If we consider now  $f = s - x(s)$  with  $x \in F^\tau$  not necessarily constant, then, away from a discrete subset of  $\mathbb{C}$  depending on  $x$ , the function

$$\Gamma(s - x(s))$$

is well defined and holomorphic. By the above observations, it certainly generates the kernel of  $\phi(f)$ . This already allows to compute the kernel of  $\phi(f)$  when  $f = \prod_{i=1}^d (s - x_i(s))$ , where  $x_i \in F^\tau$  are distinct: the kernel is generated by the functions

$$\Gamma(s - x_1(s)), \dots, \Gamma(s - x_d(s)),$$

which are linearly independent over  $F^\tau$ . In fact, by using arguments as in the proof of Proposition 16, we get the following simple improvement. The details of the proof are left to the reader.

**Proposition 19** *Let  $f = \prod_{i=1}^d (s - x_i(s))^{k_i}$  be a monic polynomial of  $F^\tau[s]$  of degree  $\sum_{i=1}^d k_i$  such that the roots  $x_i$  are distinct elements of  $F^\tau$ . Then, the kernel of  $\phi(f)$  is spanned by the  $F^\tau$ -linearly independent functions*

$$\Gamma(s - x_1(s)), \dots, \Gamma^{(k_1-1)}(s - x_1(s)), \dots, \Gamma(s - x_d(s)), \dots, \Gamma^{(k_d-1)}(s - x_d(s)),$$

*holomorphic on the complement in  $\mathbb{C}$  of a discrete set depending on  $f$ .*

The above proposition can be applied when  $f$  splits as a product of linear polynomials but cannot be applied in the opposite situation, when  $f$  is irreducible as a polynomial in  $s$ . If  $f \in F^\tau[s]$  is irreducible, there exists a unique irreducible  $g \in F^\tau[X]$  such that  $g(s) = f$ . Assuming further that the roots  $x_i(s)$  of  $g$  are in  $F^{\text{per}}$ , the functions  $\Gamma(s - x_i(s))$  are well defined on an open subset of  $\mathbb{C}$  as above, but there is no reason that they are in the kernel of  $\phi(f)$ .

However,  $\tau$  induces a permutation  $\sigma$  of the  $x_i$ . Let  $t_1, \dots, t_d$  be independent variables such that  $d = \deg f$  and such that  $\tau t_i = t_i$  for all  $i$ . If we set

$$G = G(s, t_1, \dots, t_d) = \sum_i \Gamma(s - t_i),$$

we have

$$\begin{aligned} (\tau G)|_{t_i \mapsto x_i(s)} &= \sum_i \Gamma(s + 1 - x_i(s + 1)) \\ &= \sum_i \Gamma(s + 1 - x_{\sigma(i)}(s)) \\ &= \tau(G|_{t_i \mapsto x_i(s)}). \end{aligned}$$

Hence,

$$\begin{aligned} \phi(f)\left(\sum_i \Gamma(s - x_i(s))\right) &= (\phi(f)G)|_{t_i \mapsto x_i(s)} \\ &= \left(\sum_i f(t_i) \Gamma(s - t_i)\right)|_{t_i \mapsto x_i(s)} \\ &= 0 \end{aligned}$$

and we get the following Proposition.

**Proposition 20** *Let  $f \in F^\tau[s]$  be irreducible of degree  $d$ , such that all its roots  $x_i(s)$  are in  $F^{\text{per}}$ . Then, the function*

$$\sum_i \Gamma(s - x_i(s)),$$

*defined on the complement in  $\mathbb{C}$  of a discrete subset depending on  $f$ , is a non-trivial element of the kernel of  $\phi(f)$ .*

## 2.5 The basic functional relations

So far, we have considered separately the Carlitz's modules associated to individual data  $(\mathcal{K}, \tau, \vartheta)$ , but we may well compare different structures over the same field  $\mathcal{K}$ . For example, we can choose Carlitz's module associated to the difference field  $(\mathcal{K}, \tau^n)$  with  $\vartheta$  fixed and  $n$  varying, or vary  $\vartheta$  while fixing  $(\mathcal{K}, \tau)$ . Here, we show that the Akhiezer-Baker functions corresponding to these choices satisfy functional relations that in the framework of Example (1) are at the origin of the classical functional relations of Euler's gamma function.

### 2.5.1 Multiplication relations

We compare the structures of the Carlitz modules  $\phi_{\mathcal{K}, \tau^n, \vartheta}$  and  $\phi_{\mathcal{K}, \tau, \vartheta}$  through their Akhiezer-Baker functions. From now on, we assume that  $\mathcal{K}$  is large enough to contain all the coherent sequences associated to the various Carlitz's modules  $\phi_{\mathcal{K}, \tau, \vartheta}$  that we are considering; this happens if, for instance,  $\mathcal{K}$  is existentially closed. Let us notice that if  $F$  is a non-zero solution of the difference equation

$$\tau F = AF \quad (9)$$

for some  $A \in \mathcal{K}^\times$ , then

$$\tau^n F = (\tau^{n-1} A)(\tau^{n-2} A) \cdots (\tau A) AF. \quad (10)$$

Let us now suppose that there exists a solution  $G \in \mathcal{K}^\times$  of the difference equation

$$\tau^n G = AG. \quad (11)$$

Then, for all  $i = 0, \dots, n-1$ ,  $\tau^n(\tau^i G) = (\tau^i A)(\tau^i G)$  so that the product

$$H = G(\tau G) \cdots (\tau^{n-1} G)$$

is again solution of (10). The ratio of any two non-zero solutions of the equation (10) is in  $\mathcal{K}^{\tau^n}$ . Therefore,

$$F = \lambda H, \quad (12)$$

for some  $\lambda \in \mathcal{K}^{\tau^n}$ .

The above arguments hold for any choice of  $A \in \mathcal{K}^\times$ . We now focus on the case  $A = \vartheta - t$ . In this case, any solution  $F$  of (9) is a multiple by an element of  $\mathcal{K}^\tau[[t]]$  of an Akhiezer-Baker function  $\omega_{\mathcal{K}, \tau, \vartheta, \Xi}$  associated to a coherent sequence  $\Xi$ .

On the other hand, any solution  $G$  of (11) is a multiple by an element of  $\mathcal{K}^{\tau^n}[[t]]$  of an Akhiezer-Baker function  $\omega_{\mathcal{K}, \tau^n, \vartheta, \Xi'}$  associated to a coherent sequence  $\Xi'$ . Therefore, looking at (12) we obtain the following proposition.

**Proposition 21 (Multiplication relation for Akhiezer-Baker functions)** *Consider a coherent sequence  $\Xi$  for  $\phi_{\mathcal{K}, \tau, \vartheta}$  and a coherent sequence  $\Xi'$  for  $\phi_{\mathcal{K}, \tau^n, \vartheta}$ ,  $n$  being a positive integer. If we set  $\omega = \omega_{\mathcal{K}, \tau, \vartheta, \Xi}$  and  $\omega' = \omega_{\mathcal{K}, \tau^n, \vartheta, \Xi'}$  for the corresponding Akhiezer-Baker functions, then, there exists a non-zero element  $\lambda \in \mathcal{K}^{\tau^n}[[t]]$  such that*

$$\omega = \lambda(\tau^{n-1}\omega')(\tau^{n-2}\omega') \cdots (\tau\omega')\omega'.$$

**Remark 22** The well known Gauss' *multiplication formulas* for Euler's gamma function can be deduced by first applying Proposition 21 and then computing the function  $\lambda$  by means of Stirling asymptotic formula

$$\Gamma(s) \sim \sqrt{2\pi} s^{s-1/2} e^{-s}, \quad |\arg(s)| < \pi. \quad (13)$$

An alternative way to prove this property is to use the digamma function by computing a logarithmic derivative, observing that the functions

$$G(s) = \sum_{k=0}^{N-1} \psi\left(s + \frac{k}{N}\right)$$

(for fixed  $N > 0$ ) and  $N\psi(Ns)$  are both solutions of the  $\tau$ -difference equation

$$\tau X - X = \sum_{k=0}^{N-1} \frac{N}{Ns + k},$$

so that they differ by an element of  $F^\tau$ . The details are left to the reader. Notice also that the function  $\lambda$  of Proposition 21 depends on  $t$ .

## 2.5.2 Cyclotomic relations

Let  $\zeta \in \mathcal{K}^\tau$  be a root of unity of order  $n > 0$ . We can compare the Akhiezer-Baker functions of the data  $(\mathcal{K}, \tau, \vartheta \zeta^i)$ , ( $i = 0, \dots, n-1$ ) and  $(\mathcal{K}, \tau, \vartheta^n)$  by using the obvious identity:

$$\prod_{i=0}^{n-1} (t - \zeta^i \vartheta) = t^n - \vartheta^n. \quad (14)$$

Indeed, let us choose suitable coherent sequences  $\Xi_0, \dots, \Xi_{n-1}$  and  $\Xi$  associated respectively to the Carlitz's modules  $\phi_{\mathcal{K}, \tau, \vartheta}, \phi_{\mathcal{K}, \tau, \zeta \vartheta}, \dots, \phi_{\mathcal{K}, \tau, \zeta^{n-1} \vartheta}$  and  $\phi_{\mathcal{K}, \tau, \vartheta^n}$ . Then, the corresponding Akhiezer-Baker functions satisfy:

$$\begin{aligned} \tau \omega_{\mathcal{K}, \tau, \zeta^i \vartheta, \Xi_i}(t) &= (t - \zeta^i \vartheta) \omega_{\mathcal{K}, \tau, \zeta^i \vartheta, \Xi_i}(t), \quad i = 0, \dots, n-1, \\ \tau \omega_{\mathcal{K}, \tau, \vartheta^n, \Xi}(t^n) &= (t^n - \vartheta^n) \omega_{\mathcal{K}, \tau, \vartheta^n, \Xi}(t^n). \end{aligned}$$

Multiplying term-wise the first  $n$  identities, we obtain the following proposition.

**Proposition 23 (Cyclotomic relations for Akhiezer-Baker functions)** *In the above notations, there exists a non-zero element  $\mu \in \mathcal{K}^\tau[[t]]$  such that*

$$\prod_{i=0}^{n-1} \omega_{\mathcal{K}, \tau, \zeta^i \vartheta, \Xi_i}(t) = \mu \omega_{\mathcal{K}, \tau, \vartheta^n, \Xi}(t^n).$$

The *reflection formula* for Euler's gamma function is not directly affiliated to cyclotomic relations. The appropriate framework to look at it seems to be that of the *adjoint* of Carlitz's module.



### 2.5.3 Adjunction relations

Here, we look at the Carlitz module  $\phi_{\mathcal{K}, \tau^{-1}, \vartheta}$ . This is different from the modules analyzed above because its image is in the new ring of formal series  $\mathcal{K}[[\tau^{-1}]]$ , but it can be handled in a similar way. Given a coherent sequence  $\Xi$  for  $\phi_{\mathcal{K}, \tau, \vartheta}$ , and setting  $\psi = \frac{1}{\tau \omega_{\mathcal{K}, \tau, \vartheta, \Xi}}$ , from

$$\frac{1}{\tau \omega_{\mathcal{K}, \tau, \vartheta, \Xi}} = \frac{1}{(t + \vartheta) \omega_{\mathcal{K}, \tau, \vartheta, \Xi}}$$

we deduce

$$\tau^{-1} \psi = (t + \vartheta) \psi.$$

By Proposition 13, there exists a coherent sequence  $\Xi'$  for  $\phi_{\mathcal{K}, \tau^{-1}, \vartheta}$ , the *adjoint sequence*, such that

$$\omega_{\mathcal{K}, \tau^{-1}, \vartheta, \Xi'} = \frac{1}{\tau \omega_{\mathcal{K}, \tau, \vartheta, \Xi}}.$$

Additionally, in the special case of Example (1), where the difference field is  $(F, \tau)$  and  $\vartheta = s$ , we can choose  $\Xi$  as in Proposition 15 so that  $\omega_{\mathcal{K}, \tau, \vartheta, \Xi} = \Gamma(s - t)$ . We observe that, if  $\tau_t$  is the  $F$ -linear operator (on a suitable space of functions of the variable  $t$ ) which sends  $t$  to  $t + 1$ , then

$$\tau(s - t) = \tau_t^{-1}(s - t). \quad (15)$$

This means that  $\Gamma(t - s)$  is an Akhiezer-Baker function for  $\phi_{\mathcal{K}, \tau^{-1}, \vartheta}$  and summing everything together,

$$\Gamma(t - s) = \frac{\nu}{\Gamma(s - t + 1)},$$

for some  $\nu \in \mathcal{K}^{\tau^2}[[t]]$ . Of course these arguments are covered by simply verifying that the function  $\Gamma(s)\Gamma(1 - s)$  is periodic of period 2, which is trivial, but our discussion helps in observing that the crucial point is (15), which does not hold for general difference fields  $\mathcal{K}$ . In the case of Example (2), this identity is replaced by

$$\tau(\vartheta - t) = \tau_t^{-1}(\vartheta - t)^q,$$

where  $\tau_t$  is  $\mathbb{C}_\infty$ -linear such that  $\tau_t t = t^q$ . We do not know whether this can be used to exhibit some kind of adjunction relation for the Akhiezer-Baker functions in this case, but multiplication relations and cyclotomic relations in this case hold and will be described in Section 3.2.

## 2.6 Exponential and logarithm

We introduce now another important tool of the theory, the *exponential* and the *logarithm* series associated to the data  $(\mathcal{K}, \tau, \vartheta)$ . These will be elements of  $\mathcal{K}[[\tau]]$  and we will see that sometimes and quite naturally, it is possible to associate to these formal series certain operators.

**Proposition 24** *For any given inversive difference field  $(\mathcal{K}, \tau)$  and  $\vartheta$  as above, there exist, unique, two series  $E = E_{\mathcal{K}, \tau, \vartheta}, L = L_{\mathcal{K}, \tau, \vartheta} \in \mathcal{K}[[\tau]]$  with the following properties.*

1. *The series are normalized, that is,  $E = \tau^0 + \dots, L = \tau^0 + \dots$ .*
2. *The series are one inverse of the other for the product rule of  $\mathcal{K}[[\tau]]$ :  $EL = LE = \tau^0$ .*

3. For all  $z \in \mathcal{K}$ , we have  $\phi(z)E = Ez$ .

4. For all  $z \in \mathcal{K}$ , we have  $L\phi(z) = zL$ .

*Proof.* It follows from an elementary computation. The coefficients of  $E$  can be inductively computed from the identity  $\phi(\vartheta)E = (\vartheta\tau^0 - \tau)E = E\vartheta$ :

$$E = \sum_{n \geq 0} d_n^{-1} \tau^n,$$

where  $d_0 = 1$  and

$$d_n = (\vartheta - \tau^n \vartheta) \tau d_{n-1}, \quad n \geq 1. \quad (16)$$

Similarly, the series  $L$

$$L = \sum_{n \geq 0} l_n^{-1} \tau^n$$

is uniquely determined by the recursion

$$l_n = (\tau^n \vartheta - \vartheta) l_{n-1}, \quad n \geq 1, \quad (17)$$

with  $l_0 = 1$ . Then, one notices, for  $f \in \mathcal{K}$ , the identity  $\phi(f) = EfL$ .  $\square$

In the case of Example (1), where the difference field is  $(F, \tau)$ , one immediately verifies that

$$E = \sum_{n \geq 0} (-1)^n \frac{\tau^n}{n!}, \quad L = \sum_{n \geq 0} \frac{\tau^n}{n!}.$$

More generally, we will need the *modified logarithm*. Let  $x$  be a complex number. The modified logarithm  $L_x$  is the element of  $F[[\tau]]$ :

$$L_x = \sum_{n \geq 0} \frac{x^n \tau^n}{n!}.$$

Obviously,  $E = L_{-1}$ ,  $L = L_1$  and  $L_x L_{-x} = L_{-x} L_x = \tau^0$ . Another useful identity, holding in  $F[[\tau]]$ , is

$$L_x s = \phi_x(s) L_x,$$

where  $\phi_x(s) = 1 + x\tau$ .

### 3 Link with Example (2): Anderson-Thakur's functions

In the case of Example (2), where by convention we have chosen (6) to define Carlitz's module (in particular,  $\phi_{\mathbb{C}_\infty, \tau, \vartheta}(\vartheta) = \vartheta + \tau$ ), the coefficients  $d_n, l_n$  appearing in the expansions of the series  $E_{\mathbb{C}_\infty, \tau, \vartheta}, L_{\mathbb{C}_\infty, \tau, \vartheta}$  are polynomials of  $\vartheta$  and are given by the following formulas:

$$\begin{aligned} d_n(\vartheta) &= (\vartheta^{q^n} - \vartheta^{q^{n-1}})(\vartheta^{q^n} - \vartheta^{q^{n-2}}) \cdots (\vartheta^{q^n} - \vartheta), \\ l_n(\vartheta) &= (-1)^n (\vartheta^{q^n} - \vartheta)(\vartheta^{q^{n-1}} - \vartheta) \cdots (\vartheta^q - \vartheta). \end{aligned} \quad (18)$$

From now on, we shall write, when dealing with Example (2),

$$\exp_{\tau,\vartheta} = E_{\mathbb{C}_\infty,\tau,\vartheta}, \quad \log_{\tau,\vartheta} = L_{\mathbb{C}_\infty,\tau,\vartheta}.$$

When  $\vartheta = \theta$ , we will write  $\exp = \exp_{\tau,\theta}$ ,  $\log = \log_{\tau,\theta}$ . The properties of the latter functions are described in detail in Goss' book [14] and in Thakur's book [26]. But when  $|\vartheta| > 1$ , the arguments of these authors can be easily transposed to the more general setting of the functions  $\exp_{\tau,\vartheta}$ ,  $\log_{\tau,\vartheta}$ . This follows from the elementary observation that we then have, in (18),  $|d_n(\vartheta)| = |\vartheta|^{nq^n}$  and  $|l_n(\vartheta)| = |\vartheta|^{\frac{q(q^n-1)}{q-1}}$  for all  $n \geq 0$  (see [14, Sections 3.2, 3.4]). So there is no surprise in finding the next Theorem, which specializes in well known results if  $\vartheta = \theta$ . The original description involving the *old* Carlitz module when  $\vartheta = \theta$  can be found in [8].

**Theorem 25** *Assuming that  $\vartheta \in \mathbb{C}_\infty$  is such that  $|\vartheta| > 1$ , The series  $\exp_{\tau,\vartheta} \in \mathbb{F}_q(\vartheta)[[\tau]]$  defines an entire surjective  $\mathbb{F}_q$ -linear function, again denoted by  $\exp_{\tau,\vartheta}$ :*

$$\exp_{\tau,\vartheta} : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$$

by  $\exp_{\tau,\vartheta}(z) = \sum_{i \geq 0} d_i(\vartheta)^{-1} z^{q^i}$ . The kernel of this function is a discrete, free  $\mathcal{A}$ -submodule of  $\mathbb{C}_\infty$  of rank one.

We choose, once and for all, a generator  $\tilde{\pi}_{\tau,\vartheta}$  of the kernel of  $\exp_{\tau,\vartheta}$ . It can be proved (see [14] for the case  $\vartheta = \theta$ , the general case can be proved by following the same methods) that, up to a choice of a  $(q-1)$ -th root of  $-\vartheta$ ,

$$\tilde{\pi}_{\tau,\vartheta} = \vartheta(-\vartheta)^{\frac{1}{q-1}} \prod_{i=1}^{\infty} (1 - \vartheta^{1-q^i})^{-1} \in (-\vartheta)^{\frac{1}{q-1}} \mathbb{F}_q((\vartheta^{-1})). \quad (19)$$

If  $\vartheta = \theta$ , we shall write  $\tilde{\pi}$  instead of  $\tilde{\pi}_{\tau,\vartheta}$ . This is the so-called *fundamental period* of Carlitz's module in the framework of Example (2) (there are  $q-1$  such choices). It is often compared to the complex number  $2\pi i$  due to several analogies connecting the multiplicative group  $\mathbb{G}_m$  and the Carlitz's module. In fact, we have just seen that there are *infinitely many* fundamental periods, depending on the various choices of  $\vartheta$  we can do to fix particular maps  $\phi_{\mathbb{C}_\infty,\tau,\vartheta}$ .

We can also fix  $\vartheta$  and replace  $\tau$  by  $\tau^n$  for  $n > 1$ . The above discussion applies because the formulas (18) still hold, up to replacement of  $q$  by  $q^n$ .

**Remark 26** In [15, 16] as well as in other references collected in [15], the reader can find the development of these ideas in yet another framework that we can consider as the *Example (4)*. Let us choose, following Hellegouarch, a field  $L$  and an indeterminate  $\vartheta$  transcendental over  $L$  (including the case  $L = \mathbb{F}_q$ ). Over the field of formal series  $\mathcal{K} = L((1/\vartheta))$ , we consider the  $L$ -linear endomorphism  $\tau$  determined by

$$\tau\vartheta = \vartheta^d + a_1\vartheta^{d-1} + a_2\vartheta^{d-2} + \dots$$

where  $d$  is an integer  $\geq 2$  and  $a_1, a_2, \dots$  are elements of  $L$ . We then have that  $\mathcal{K}^{\tau^n} = L$  for all  $n \geq 1$ . Then (see [15]), the series  $E$  in the framework of Carlitz module  $\phi_{\mathcal{K},\tau,\vartheta}$  induces, over a suitable field of Puiseux series containing  $\mathcal{K}$ , a well defined and  $L$ -linear function, the kernel of which is a discrete submodule, establishing a good analogue of Theorem 25. Among several other facts we will see, as a consequence of the results in Section 4, that the bound  $d \geq 2$  is best possible.

### 3.1 The Anderson-Thakur function

Since  $\mathbb{C}_\infty$  is algebraically closed, coherent sequences  $\Xi$  in the framework of Example (2) (with a choice of  $\vartheta$ ) exist in  $\mathbb{C}_\infty$  and can be easily constructed by solving iterated Artin-Schreier equations. Just as for Euler's gamma function (1), there is here a canonical choice of coherent sequence which can be made thanks to the exponential function, when  $|\vartheta| > 1$ .

Indeed, theorem 25 allows us to immediately verify that if  $|\vartheta| > 1$ ,

$$\Xi = \left( \exp_{\tau, \vartheta} \left( \frac{\tilde{\pi}_{\tau, \vartheta}}{\vartheta^{i+1}} \right) \right)_{i \geq 0}$$

is well defined and is a coherent sequence. We will call the Akhiezer-Baker function  $\omega_{\mathbb{C}_\infty, \tau, \vartheta, \Xi}$  the *Anderson-Thakur function* associated to the data  $(\mathbb{C}_\infty, \tau, \vartheta)$ , assigning to it the simpler notation  $\omega_{\mathbb{C}_\infty, \tau, \vartheta}$ . It can be defined, equivalently, by the formulas:

$$\omega_{\mathbb{C}_\infty, \tau, \vartheta}(t) = \exp_{\tau, \vartheta} \left( \frac{\tilde{\pi}_{\tau, \vartheta}}{\vartheta - t} \right) = \sum_{i \geq 0} \frac{\tilde{\pi}_{\tau, \vartheta}^{q^i}}{d_i(\vartheta)(t - \vartheta^{q^i})}, \quad (20)$$

where  $\exp_{\tau, \vartheta}$  must be considered as an  $\mathbb{F}_q((t))$ -linear operator (cf. [22]). From now on, we will also denote by  $\omega$  the *original Anderson-Thakur function* appearing in [4], thus associated with the choice  $\vartheta = \theta$ . We recall that  $\omega_{\mathbb{C}_\infty, \tau, \vartheta}$  satisfies, after Proposition 13, the identities:

$$\phi_{\mathbb{C}_\infty, \tau, \vartheta}(a)\omega_{\mathbb{C}_\infty, \tau, \vartheta}(t) = a(t)\omega_{\mathbb{C}_\infty, \tau, \vartheta}(t), \quad a \in \mathcal{A}. \quad (21)$$

The next proposition contains the fundamental properties of the functions  $\omega_{\mathbb{C}_\infty, \tau, \vartheta}$ . They are simple generalizations of well known properties of the function  $\omega = \omega_{\mathbb{C}_\infty, \tau, \theta}$  that can be found in [3, 4, 5, 22]. The proof of the Proposition uses the formulas (20) and does not offer additional difficulty so it will be omitted.

**Proposition 27** *Let  $\vartheta$  be an element of  $\mathbb{C}_\infty$  such that  $|\vartheta| > 1$ . We have the following properties.*

1. *The formal series  $\omega_{\mathbb{C}_\infty, \tau, \vartheta}(t) \in \mathbb{C}_\infty[[t]]$  converges for  $t \in \mathbb{C}_\infty$  such that  $|t| < |\vartheta|$  to a rigid analytic function.*
2. *The function  $1/\omega_{\mathbb{C}_\infty, \tau, \vartheta}$  extends to an entire rigid analytic function  $\mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$  with zeros precisely located at the elements  $\vartheta, \vartheta^q, \dots$*
3. *The function  $\omega_{\mathbb{C}_\infty, \tau, \vartheta}$  extends to a meromorphic function  $\mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$  with no zeros, and simple poles at  $\vartheta, \vartheta^q, \dots$  of residues  $-\tilde{\pi}_{\tau, \vartheta}, -\frac{\tilde{\pi}_{\tau, \vartheta}^q}{d_1(\vartheta)}, \dots$ , where the coefficients  $d_i(\vartheta)$  are defined in (18).*

### 3.2 Functional relations

The results of Section 2.5 can be applied to describe the multiplication relations and the cyclotomic relations for the function  $\omega$ . Following [22], we observe that

$$\omega(t) = (-\theta)^{\frac{1}{q-1}} \prod_{i > 0} \left( 1 - \frac{t}{\theta^{q^i}} \right)^{-1},$$

with a suitable choice of  $(q-1)$ -th root of  $-\theta$ . More generally, if  $\zeta \in \mathbb{F}_q^\times$  is such that  $\zeta^n = 1$  and  $\zeta^k \neq 1$  for  $0 < k < n$ , we have

$$\begin{aligned}\omega_{\mathbb{C}_\infty, \tau^n, \theta}(t) &= (-\theta)^{\frac{1}{q^n-1}} \prod_{i>0} \left(1 - \frac{t}{\theta^{q^{ni}}}\right)^{-1}, \\ \omega_{\mathbb{C}_\infty, \tau, \zeta\theta}(t) &= (-\zeta\theta)^{\frac{1}{q-1}} \prod_{i>0} \left(1 - \frac{t}{\zeta\theta^{q^i}}\right)^{-1}, \\ \omega_{\mathbb{C}_\infty, \tau, \theta^n}(t) &= (-\theta^n)^{\frac{1}{q-1}} \prod_{i>0} \left(1 - \frac{t}{\zeta\theta^{nq^i}}\right)^{-1}.\end{aligned}$$

These formulas immediately imply that  $\lambda = \mu = 1$  in Propositions 21 and 23. More explicitly, we derive the following Proposition.

**Proposition 28** *With  $\zeta, n$  as above, the following formulas hold.*

1. *Multiplication relations.*

$$\omega_{\mathbb{C}_\infty, \tau, \theta}(t) = \prod_{i=0}^{n-1} \tau^i \omega_{\mathbb{C}_\infty, \tau^n, \theta}(t),$$

2. *Cyclotomic relations.*

$$\omega_{\mathbb{C}_\infty, \tau, \theta^n}(t^n) = \prod_{i=0}^{n-1} \omega_{\mathbb{C}_\infty, \tau, \zeta^i \theta}(t).$$

The latter relations are very similar to the *reflection formulas* for the *geometric gamma function* as in Thakur's article [25]. It is for this reason that in [22], we have mentioned them as the “analogues of the reflection formula” for Euler's gamma function.

### 3.3 The function $\omega$ and the torsion

Anderson-Thakur's function  $\omega$  interpolates many torsion elements for the Carlitz module in the framework of Example (2), just as the gamma function in Proposition 19. Indeed, the following Proposition holds.

**Proposition 29** *Let  $a$  be irreducible of degree  $d$  in  $\mathbb{F}_q[\theta]$ . Let  $\xi_1, \dots, \xi_d$  be the roots of  $a$  in  $\mathbb{F}_{q^d}$ . The following identity holds:*

$$\omega(\xi_1) + \dots + \omega(\xi_d) = \exp\left(\frac{\tilde{\pi}a'}{a}\right),$$

where  $a'$  indicates the first derivative of  $a$  with respect to  $\theta$ .

*Proof.* We have the following identities:

$$\begin{aligned}
\sum_{j=1}^d \omega(\xi_j) &= \sum_{n \geq 0} \exp\left(\frac{\tilde{\pi}}{\theta^{n+1}}\right) (\xi_1^n + \cdots + \xi_d^n) \\
&= \sum_{n \geq 0} \exp\left(\frac{\tilde{\pi}}{\theta^{n+1}} (\xi_1^n + \cdots + \xi_d^n)\right) \\
&= \exp\left(\tilde{\pi} \sum_{j=1}^d \sum_{n \geq 0} \theta^{-n-1} \xi_j^n\right) \\
&= \exp\left(\tilde{\pi} \sum_{j=1}^d \frac{1}{\theta - \xi_j}\right) \\
&= \exp\left(\frac{\tilde{\pi} a'}{a}\right).
\end{aligned}$$

The essential step is that  $\sum_j \xi_j \in \mathbb{F}_q$  so that we can use  $\mathbb{F}_q$ -linearity of  $\exp$ .  $\square$

**Remark 30** More generally, it is possible to show the following property, independently noticed by B. Anglès. Let  $L$  be the compositum in  $\mathbb{C}_\infty$  of the field  $\mathbb{F}_q^{\text{alg}}$  and the various torsion subfields  $K(\ker(\phi(\mathfrak{p})))$  for  $\mathfrak{p}$  monic irreducible polynomial of  $A$ . Then,  $L$  is equal to the field generated over  $\mathbb{F}_q^{\text{alg}}$  by the values  $\omega(\xi)$ , with  $\xi$  varying in  $\mathbb{F}_q^{\text{alg}}$ . By [22, Corollary 5], if  $\xi$  is in  $\mathbb{F}_{q^d} \setminus \mathbb{F}_{q^{d-1}}$ , then the set of solutions in  $\mathbb{C}_\infty$  of the equation

$$X^{q^d-1} = (\xi - \theta^{q^d}) \cdots (\xi - \theta)$$

is the subset

$$\mathbb{F}_{q^d}^\times \omega(\xi).$$

B. Anglès also pointed out that the function  $\omega$  can be understood as an “universal Gauss-Thakur sum”.

## 4 Back to Example (1): the series $L_x$ as operators

In the specific case of Example (1), we want to investigate the nature of the functions  $E, L$ . For this, we need appropriate spaces on which these operators act, but the problem to find natural ones seems to be difficult, unlike the case of Example (2). Here, we propose an algebra  $\mathbb{B}$  of functions over which  $L_x$  induces an automorphism for all  $x \in \mathbb{C}$ . This is an important difference compared with the case of Example (2) with, say,  $\vartheta = \theta$ , where the exponential function  $\exp$  has non-trivial kernel (a free  $A$ -module of rank one, generated by Carlitz’s period  $\tilde{\pi}$ ).

**Definition 31** Let us consider the ring  $\mathcal{B}$  of functions defined and continuous for  $s$  with  $\Re(s) \geq 0$ , holomorphic for  $\Re(s) > 0$ , such that, in any horizontal half-strip

$$\{z \in \mathbb{C}; 0 < \alpha_0 < \Re(z), \beta_0 < \Im(z) < \beta_1\},$$

the function  $|f(z)|$  is dominated by  $c_1 e^{c_2|z|}$ , where  $c_1, c_2$  are positive real numbers depending on the half-strip.

The ring  $\mathcal{B}$  is an integral domain and is an algebra over the ring of entire functions which are periodic of period 1. The  $F^\tau$ -algebra  $\mathbb{B} = \mathcal{B} \otimes F^\tau$  will be sometimes called the *algebra of test functions*.

The algebra  $\mathbb{B}$  is endowed with a semi-norm  $\|\cdot\|$  in the following way. Let us first define  $\|\cdot\|$  over the algebra  $\mathcal{B}$ . Let  $f$  be an element of  $\mathcal{B}$ ; we set:

$$\|f\| = \inf\{c > 0, \lim_{\alpha \rightarrow \infty} f(\alpha)c^{-\alpha} = 0\}.$$

The above, is a semi-norm. It is not a norm; in particular, functions  $f$  which are non-constant with  $\|f\| = 0$  exist, like the function  $1/\Gamma(s)$ . Since for  $f, g \in \mathcal{B}$  we have  $\|f + g\| \leq \max\{\|f\|, \|g\|\}$ , by setting  $\|h\| = 1$  for all  $h \in F^\tau \setminus \{0\}$ , there exists a unique semi-norm  $\|\cdot\|$  on  $\mathbb{B}$  extending the previous one. The automorphism  $\tau$  of  $F$  induces an endomorphism of  $\mathbb{B}$ . On the other hand, by the fact that polynomials with complex coefficients belong to  $\mathbb{B}$ , Carlitz's module  $\phi_{F, \tau, s}$  induces an action of  $\mathcal{A}$  over  $\mathbb{B}$ .

**Remark 32** The algebra  $\mathcal{B}$  contains, as a subring, the ring  $\mathcal{E}$  of the *entire functions of exponential type* <sup>(5)</sup>. If  $f \in \mathcal{E}$ , then  $\|f\|$  is less than the exponential of the *type* of  $f$ . We recall that Siegel's  $E$ -functions are examples of entire functions of exponential type. It can also be proved that the function  $1/\Gamma(s)$  and the function  $(1-s)\zeta(s)$  belong to  $\mathcal{E}$ . But notice that no one of the functions  $\Gamma(s+z)$ ,  $z \in \mathbb{C}$  belongs to  $\mathbb{B}$ , as Stirling asymptotic estimate indicates.

We now look at  $E$  and, more generally, at  $L_x$ , as operators on  $\mathbb{B}$ .

**Proposition 33** *For all  $x \in \mathbb{C}$ ,  $L_x$  defines an  $F^\tau$ -linear automorphism of  $F^\tau$ -vector spaces*

$$L_x : \mathbb{B} \rightarrow \mathbb{B}$$

*of inverse  $L_{-x}$ . For all  $f \in \mathbb{B}$  and  $x \in \mathbb{C}$ ,*

$$\|L_x f\| \leq \|f\|.$$

*Proof.* First of all, for any  $x \in \mathbb{C}$  and  $f \in \mathbb{B}$ ,  $L_x f$  is clearly well defined. Moreover, if  $|f(\alpha + i\beta)| \leq c_1 e^{c_2 \alpha}$  for  $\alpha$  big enough, then

$$|L_x f(\alpha + i\beta)| \leq c_1 e^{e^{c_2}|x|} e^{c_2 \alpha},$$

so that  $L_x f \in \mathbb{B}$  and  $\|L_x f\| \leq \|f\|$ .

The function  $L_x$  is  $F^\tau$ -linear because on one side it is additive, and on the other side, if  $f \in \mathbb{B}$  and  $\lambda \in F^\tau$ , then:

$$\begin{aligned} L_x(\lambda f) &= \sum_{n \geq 0} x^n \frac{\tau^n(\lambda f)}{n!} \\ &= \lambda \sum_{n \geq 0} x^n \frac{\tau^n(f)}{n!} \\ &= \lambda L_x(f). \end{aligned}$$

---

<sup>5</sup>An entire functions of exponential type is an entire function  $f : \mathbb{C} \rightarrow \mathbb{C}$  such that

$$|f(s)| \leq c_1 e^{c_2|s|}, \quad \text{for all } s \in \mathbb{C},$$

for some  $c_1, c_2 > 0$ .

By the fact that  $L_x L_{-x} = L_{-x} L_x = \tau^0$  we see that if  $f$  is such that  $L_x f = 0$ , then  $f = 0$  so  $L_x$  is bijective. In particular,  $L = L_1$  and  $E = L_{-1}$  induce two  $F^\tau$ -linear automorphisms of  $\mathbb{B}$ , one inverse of the other.  $\square$

**Remark 34** We can also compare the above observations with Remark 26 by considering, instead of the field  $\mathcal{K} = F$ , the field  $\mathcal{K} = \mathbb{C}((s^{-1}))$  which is also endowed with the automorphism  $\tau$  defined by  $\tau(s) = s+1$  (so we are in the case  $d = 1$ ). The reader can verify that the corresponding functions  $L_x$  are all well defined  $\mathbb{C}$ -automorphisms.

## 4.1 Compatibility with Mellin's transform

The use of the algebra  $\mathbb{B}$  has some advantages, but, as we pointed out,  $\Gamma$  does not count among its elements. However, the Mellin transform is well defined on  $\mathcal{B}$ , the operators  $L_x$  have good compatibility with it, and the gamma function is the Mellin transform of  $e^{-t}$ .

We focus on the structure of  $\mathbb{C}$ -vector space of  $\mathcal{B}$ , that we filter according to the types. For  $\alpha$  real, let  $\mathcal{B}_\alpha$  be the vector space of functions  $f$  of  $\mathcal{B}$  such that  $|f(t)| \leq c_1 e^{-t\alpha}$  for  $t$  big enough. We have, for  $\alpha \leq \beta$ ,  $\mathcal{B}_\beta \subset \mathcal{B}_\alpha$ .

If  $f(t)$  is a continuous function of the real variable  $t \in ]0, \infty[$  such that  $f$  is right continuous at 0 and such that  $f(t) = O(e^{-\alpha t})$  for  $\alpha > 0$  as  $t \rightarrow \infty$ , then, the integral (the Mellin transform of  $f$ )

$$F(s) = \int_0^\infty t^{s-1} f(t) dt$$

converges in the half-plane  $\Re(s) > 0$ . For  $\alpha > 0$ , let us denote by  $V_\alpha$  the image under the Mellin transform of the space  $\mathcal{B}_\alpha$ . We also recall that the Mellin transform, as an operator on a space  $\mathcal{B}_\alpha$  for fixed  $\alpha$ , is injective.

**Lemma 35** *Let  $f$  be an element of  $\mathcal{B}_\alpha$  with  $\alpha > 0$ , so that its Mellin transform  $F(s)$  defines an holomorphic function for  $\Re(s) > 0$ . Then, if  $x$  is a complex number such that*

$$|x| < \alpha,$$

*the evaluation of the formal series  $L_x \in \mathbb{Q}[x][[\tau]]$  at  $F$  is well defined and is the Mellin transform of the function  $e^{tx} f(t) \in \mathcal{B}_{\alpha-|x|}$ .*

*Proof.* Let  $\sigma$  be the real part of  $s$ . There exists a constant  $c > 0$  such that, for  $n \geq 0$ ,

$$\left| \frac{x^n}{n!} \int_0^\infty t^{s+n-1} f(t) dt \right| \leq c \alpha^{-\sigma} \left| \frac{x}{\alpha} \right|^n \frac{\Gamma(n+\sigma)}{n!}.$$

Under the above conditions and setting  $X = |x/\alpha|$ , the series

$$\sum_{n \geq 0} X^n \frac{\Gamma(n+\sigma)}{n!}$$

converges to the function  $\frac{\Gamma(\sigma)}{(1-X)^\sigma}$  and the following transformations hold:

$$\begin{aligned} L_x \int_0^\infty t^{s-1} f(t) dt &= \int_0^\infty L_x(t^{s-1}) f(t) dt \\ &= \int_0^\infty t^{s-1} e^{tx} f(t) dt. \end{aligned}$$



□

*Examples.* We have  $\Gamma \in V_1$ . Therefore,

$$L_x \Gamma(s) = \frac{\Gamma(s)}{(1-x)^s} \in V_{1-|x|}$$

for all  $x$  complex such that  $|x| < 1$ . In the next table we provide a list of useful identities involving the operators  $L_x$ .

| $f$                             | $L_x(f)$                               | condition                                |
|---------------------------------|--|--|
| $L_{-x}(f)$                     | $f$                                    |  |
| $t^s f$                         | $e^{xt} t^s L_{xt}(f(s))$              |  |
| $sf$                            | $sL_x(f) + L_x(\tau f)$                |  |
| $\tau f$                        | $\frac{d}{dx}(L_x(f))$                 |  |
| $\int_0^\infty t^{s-1} g(t) dt$ | $\int_0^\infty t^{s-1} e^{tx} g(t) dt$ | $g \in \mathcal{B}_\alpha, \alpha >  x $ |
| $Q$                             | $\sum_{i \geq 0} E_i(Q) x^i$           | $Q \in \mathbb{C}[s]$                    |
| $\Gamma(s)$                     | $(1-x)^{-s} \Gamma(s)$                 | $ x  < 1$                                |

More generally, the operators  $L_x$  can be used to construct various classical special functions. For example, we can obtain Gauss' hypergeometric functions as images of fractions involving products of the gamma function via  $L_x$ . The reader can find many other examples by just examining any textbook of special functions like [1].

**Remark 36** We recall a formula which can be deduced from Lommel's Theorem for the Bessel functions (cf. [27, p. 141]):

$$\sum_{n \geq 0} \frac{(-1)^n (z/2)^n}{n!} J_{\nu-m}(z) = 0, \quad \Re(\nu) > 0, \quad z \in \mathbb{C}^\times,$$

where  $J_\nu(z)$  denotes Bessel's function of first kind. The formula can be rewritten as

$$E^*((-z/2)^{-s} J_s(z)) = 0,$$

where  $E^* = \sum_{n \geq 0} \tau^{-n}/n!$  is the exponential series  $E$  associated to the data  $(F, \tau^{-1}, s)$  and  $z$  is fixed. In particular, the kernel of the exponential  $E^*$  is non-trivial.

**Remark 37** The author owes the following remark to D. Goss. The integral formula of the gamma function (1) can be interpreted as follows. For  $z \in \mathbb{C}$  such that  $\Re(z) > 0$ , let us consider the family of measures  $d\mu_z$  on  $\mathbb{R}_{>0}$  by  $d\mu_z = t^z dt/t$ . Then

$$\Gamma(z) = \int_{\mathbb{R}_{>0}} e^{-t} d\mu_z(t).$$

Now, in the framework of Example (2) with  $\vartheta = \theta$ , let  $\delta_a$  be the Dirac measure at  $a \in \mathbb{C}_\infty$  and consider, for a parameter  $t \in \mathbb{C}_\infty$ , the distribution

$$d\nu_t = \sum_{i \geq 0} t^i \delta_{1/\theta^{i+1}}$$

For  $|t|$  small, this converges to a measure on the ring  $\mathcal{O}$  of integers of  $K_\infty$ . Then, we have

$$\omega(t) = \int_{\mathcal{O}} \exp(\tilde{\pi}z) d\nu_t(z),$$

similar to (1). Moreover, for  $a \in A$ ,

$$\begin{aligned} a(t)\omega(t) &= \phi(a) \int_{\mathcal{O}} \exp(\tilde{\pi}z) d\nu_t(z) \\ &= \int_{\mathcal{O}} \phi(a) \exp(\tilde{\pi}z) d\nu_t(z) \\ &= \int_{\mathcal{O}} \exp(\tilde{\pi}az) d\nu_t(z), \end{aligned}$$

in analogy with the formula

$$\nu^{-s}\Gamma(s) = \int_{\mathbb{R}_{>0}} e^{-\nu t} d\mu_z(t), \quad \Re(\nu) > 0.$$

However, the analogy is only partial. In (1), the function we integrate is related to the exponential function  $e^x = \sum_{i \geq 0} x^i / i!$  of the multiplicative group  $\mathbb{G}_m$  but not to the exponential function of the generalized Carlitz module of Example (1). In fact, the function  $e^x$  cannot be naturally seen as the exponential function of a generalized Carlitz's module as described in this text, for the reason that  $\mathbb{Z}$  (the ring that here should correspond to  $\mathcal{A}$ ) is not of the form  $L[\vartheta]$  for a field  $L$  and an element  $\vartheta$  transcendental over  $L$ !

## 5 Some applications of Carlitz formalism

We return here to the framework of Example (1) and observe that some simple and classical properties of Hurwitz's zeta function

$$\zeta(z, s) = \sum_{n \geq 0} \frac{1}{(n+s)^z}, \quad \Re(z) > 1, \quad \Re(s) > 0$$

can be easily verified with the Carlitz formalism and some standard asymptotic estimates.

Riemann's integral formula

$$\zeta(z)\Gamma(z) = \int_0^\infty t^{z-1} \frac{1}{e^t - 1} dt \tag{22}$$

extends to Hurwitz zeta function by means of the integral formula:

$$\zeta(z, s)\Gamma(z) = \int_0^\infty e^{-st} t^{z-1} \frac{1}{1 - e^{-t}} dt \tag{23}$$

(Riemann's formula corresponds to the value  $s = 1$ ). This implies that  $\zeta(z, s)$ , for fixed  $s$ , extends to a meromorphic function on  $\mathbb{C}$  with, as unique singularity, a simple pole at  $z = 1$  with residue 1 (thus independent on the choice of  $s$ ).

The next question is then to look at the Laurent series expansion of  $\zeta(z, s)$  at  $z = 1$ . The coefficients are often used to define the so-called Stieltjes constants. Here is a very classical result for the constant term in this Laurent series.

**Proposition 38** *The following limit holds:*

$$\lim_{z \rightarrow 1} \zeta(z, s) - \frac{1}{z-1} = -\frac{\Gamma'(s)}{\Gamma(s)}.$$

*Proof.* This limit can be computed in a variety of ways. We propose here to use Carlitz's formalism. We begin by showing that the function

$$f(s) := \Gamma(s) \left( \lim_{z \rightarrow 1} \zeta(z, s) - \frac{1}{z-1} \right)$$

is an element of  $s^2$ -torsion.

We compute first  $\phi(s)f(s)$ , that is,  $sf(s) - f(s+1)$ . We find:

$$\begin{aligned} \phi(s)f(s) &= s \left( \Gamma(s) \left( \lim_{z \rightarrow 1} \zeta(z, s) - \frac{1}{z-1} \right) \right) - \Gamma(s+1) \left( \lim_{z \rightarrow 1} \zeta(z, s+1) - \frac{1}{z-1} \right) \\ &= s\Gamma(s) \left( \lim_{z \rightarrow 1} \zeta(z, s) - \zeta(z, s+1) \right) \\ &= \Gamma(s). \end{aligned}$$

Therefore,  $\phi(s^2)f(s) = \phi(s)\Gamma(s) = 0$ , so  $f(s)$  is an element of  $s^2$ -torsion as claimed. This means, by Proposition 15, that there exist two functions  $a(s), b(s) \in F^\tau$  such that

$$f(s) = a(s)\Gamma(s) + b(s)\Gamma'(s), \quad (24)$$

and we need show that  $a, b$  are constant, with  $a = 0$ ,  $b = -1$ . For this, an asymptotic estimate will suffice. We appeal to Euler-MacLaurin summation formula which yields, for  $\Re(z) > 1$  and  $\Re(s) > 0$ :

$$\zeta(z, s) - \frac{1}{z-1} = \frac{s^{1-z} - 1}{z-1} + \frac{s^{-z}}{2} - z \int_0^\infty \frac{t - [t] - \frac{1}{2}}{(t+s)^{z+1}} dt.$$

Now, the limit for  $z \rightarrow 1$  of  $\frac{s^{1-z}-1}{z-1}$  is well defined, being equal to  $-\log(s)$  so that we have the well defined limit

$$\lim_{z \rightarrow 1} \zeta(z, s) - \frac{1}{z-1} = -\log(s) + \frac{1}{2s} - \int_0^\infty \frac{t - [t] - \frac{1}{2}}{(t+s)^2} dt.$$

For  $s$  real tending to  $\infty$ , the left-hand side of the above identity is then asymptotically equivalent to  $-\log(s)$ . But it is easy to verify that the only element of the  $F^\tau$ -span of the functions  $1, \frac{\Gamma'(s)}{\Gamma(s)}$  with this property is precisely  $-\Gamma'(s)/\Gamma(s)$  so that  $a = 0$  and  $b = -1$ .  $\square$

**Remark 39** More generally, let  $\Psi(s, t)$  be the function

$$\Psi(s, t) = \sum_{n \geq 1} \zeta(n+1, s) t^n.$$

Then, the identity

$$\Psi(s, t) = \psi(s) - \psi(s-t)$$

holds, for  $\Re(s) > \Re(t) > 0$ . The proof uses the fact that  $\psi(s) - \Psi(s, t)$  is solution of the non-homogeneous  $\tau$ -difference equation (8), and standard asymptotical estimates. This yields well known formulas (see for instance [1, 25.11.12]):

$$\zeta(n+1, s) = \frac{(-1)^{n+1} \psi^{(n)}(s)}{n!}, \quad n \geq 1. \quad (25)$$

**Remark 40** It is also well known that the above results can be directly deduced from Weierstrass' product expansion of the gamma function. Carlitz's formalism and asymptotic estimates allow to deduce this product expansion as well; this is implicit in the enlightening introduction of Aomoto and Kita's book [6, p. 1-3].

We provide a last application of Carlitz formalism, this time involving the operator  $L_x$ . The following formula appears in [28] and in [24, p. 137 (6.6)], and generalizes a classical formula by Landau. If  $x, \alpha$  are complex numbers such that  $|x| < |\alpha|$  and  $\alpha$  is not in the set  $\{0, -1, -2, \dots\}$ , then, for  $s$  complex number different from 1,

$$\sum_{n \geq 0} \binom{s+n-1}{n} \zeta(s+n, \alpha) x^n = \zeta(s, \alpha - x).$$

In our notations, we have:

**Proposition 41** *For  $x, \alpha$  complex such that  $|x| < |\alpha|$  and  $\Re(\alpha) > 0$ , we have:*

$$L_x(\Gamma(s)\zeta(s, \alpha)) = \Gamma(s)\zeta(s, \alpha - x).$$

*Proof.* It can be deduced from (23) and an appropriate variant of Lemma 35 when  $\alpha > 0$  but it extends to complex  $\alpha$  with  $\Re(\alpha) > 0$  as well. The following intermediate identities can be easily verified:

$$\begin{aligned} L_x(\Gamma(s)\zeta(s, \alpha)) &= \int_0^\infty e^{-\alpha t} L_x(t^{s-1}) \frac{1}{1-e^{-t}} dt \\ &= \int_0^\infty e^{-\alpha t} t^{s-1} \frac{e^{tx}}{1-e^{-t}} dt \\ &= \int_0^\infty e^{-(\alpha-x)t} t^{s-1} \frac{1}{1-e^{-t}} dt \\ &= \zeta(s, \alpha - x) \Gamma(s). \end{aligned}$$

□

## 6 Link with the functions $L(\chi_t^\beta, \alpha)$ .

We end our discussion about Carlitz formalism by considering once again Example (2) and we restrict our attention to the case  $\vartheta = \theta$ . Let  $A^+$  be the set of monic polynomials of  $A = \mathbb{F}_q[\theta]$ . In [22], we have introduced, for  $\beta \geq 0$  and  $\alpha > 0$ , the series:

$$L(\chi_t^\beta, \alpha)(t) = \sum_{a \in A^+} \chi_t(a^\beta) a^{-\alpha} \in K_\infty[[t]].$$

Here, for a given polynomial  $a$  of  $A$ ,  $\chi_t(a)$  denotes the polynomial function obtained by substituting  $\theta$  by  $t$ . In fact, the series  $L(\chi_t^\beta, \alpha)$  converges for  $t \in \mathbb{C}_\infty$  with  $|t| \leq 1$  and has analytic extension to the whole  $\mathbb{C}_\infty$ -plane.

We recall, following [22], that for  $t \in \{\theta^{q^k}, k \in \mathbb{Z}\}$ ,  $L(\chi_t^\beta, \alpha)(t)$  specializes to a *Carlitz zeta value*. For example, for  $t = \theta$ , we get

$$L(\chi_t^\beta, \alpha)(\theta) = \zeta(\alpha - \beta) = \sum_{d \geq 0} \sum_{a \in A^+(d)} a^{\beta - \alpha},$$

where  $A^+(d)$  denotes the set of monic polynomials of degree  $d$ .

Also, in [22], we have stressed that special values of  $L$ -series associated to a certain Dirichlet character may also be obtained as special values of the series  $L(\chi_t^\beta, \alpha)$ , this time substituting  $t$  with a root of unity. This already gives the series  $L(\chi_t^\beta, \alpha)$  a flavor of Hurwitz's zeta function because the latter function is also used to compute  $L$ -series associated to Dirichlet characters in the domain of their analytic continuation.

There are other signs which allow to pursue the analogy with the Hurwitz's zeta function. One of the results in [22] is the following formula:

$$L(\chi_t, 1)(t) = -\frac{\tilde{\pi}}{(t - \theta)\omega(t)}. \quad (26)$$

Among several things, this formula, together with Proposition 27, allows to prove directly that  $L(\chi_t, 1)(t)$  can be analytically extended to  $\mathbb{C}_\infty$ , and provides important information on certain Carlitz's zeta values, as well as on certain trivial zeros of Goss' zeta function (see [22]). Now, this goes in the same direction of Proposition 38 and (25).

Here is a last analogy. In a work in progress, the author and R. Perkins recently considered an extension of the semi-character  $\chi_t$  to the whole  $\mathbb{C}_\infty$ -plane. In order to construct this, one appeals to Anderson-Thakur's function by setting, for  $z \in \mathbb{C}_\infty$ ,

$$\chi_t(z) = \frac{\phi(z)\omega(t)}{\omega(t)},$$

$\phi(z)$  being the operator of  $\mathbb{C}_\infty[[\tau]]$  as in Definition 1. The evaluation at  $\omega$  is possible (there is a suitable notion of radius of convergence to take into account here). In particular, if  $z = a \in A$ , we get the above semi-character by (21). Now, extending  $\tau$  to the ring  $\mathbb{C}_\infty[[t, t_1]]$  with the rule  $\tau t_1 = t_1$ , where  $t_1$  is another variable, we can as well evaluate the operator  $\log = \sum_{n \geq 0} (-1)^n l_n^{-1} \tau^n$  at a series of  $\mathbb{C}_\infty[[t, t_1]]$ . We show that

$$\log \left( \omega \sum_{a \in A^+} \chi_t(a^{-1}) \chi_{t_1}(a) \right) = \frac{-\tilde{\pi}}{t - \theta} L(\chi_{t_1}, 1), \quad (27)$$

a formula which has some similarity with Proposition 41.

**Remark 42** The theory of the generalized Carlitz's module can be extended to higher rank case in parallel with the theory of Drinfeld modules. We will not discuss it here, but again the classical theory of special functions offers abundant examples of these issues. For example, the well known *contiguity relations* for Gauss' hypergeometric functions may be viewed, after suitable normalizations as particular cases of *generalized Drinfeld modules of rank two* in the framework of Example (1). The author has already partially tracked this analogy in the recent paper [21].

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